Homework 2
MTH 869 Algebraic Topology

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Lemma 0.1 (for Example 3.8). *Let N_g be the closed nonorientable surface of genus g. The cohomology of N_g with $\mathbb{Z}_2$ coefficients is*

$$H^k = \begin{cases} \mathbb{Z}_2 & k = 0 \\ (\mathbb{Z}_2)^g & k = 1 \\ \mathbb{Z}_2 & k = 2 \\ 0 & k \geq 3 \end{cases}$$

Proof. (We use $\mathbb{Z}_n$ as shorthand for $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}^g$ as shorthand for $\oplus_{i=1}^g \mathbb{Z}$.) Via a cellular homology computation, the homology of N_g with $\mathbb{Z}$ coefficients is

$$H_k(N_g) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 & k = 1 \\ 0 & k \geq 2 \end{cases}$$

By the Universal Coefficient Theorem, the cohomology of N_g with $\mathbb{Z}_2$ coefficients is

$$H^k(N_g; \mathbb{Z}_2) \cong \operatorname{Ext}(H_{k-1}(N_g), \mathbb{Z}_2) \oplus \operatorname{Hom}(H_k(N_g), \mathbb{Z}_2)$$

Using the above equality, we can compute H^k to be as claimed. $\square$

Proposition 0.2 (for Example 3.8). *Let N_g be the closed nonorientable surface of genus g. The cup product $H^k \times H^\ell \to H^{k+\ell}$ is as follows. For $k = 0$,*

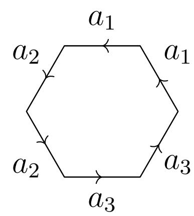
$$\smile : \mathbb{Z}_2 \times H^\ell \to H^\ell \qquad 0 \smile x = 0 \qquad 1 \smile x = x$$

When $k + \ell \geq 3$, $\smile$ is the zero map. For $k = \ell = 1$, $\smile : H^1 \times H^1 \to H^2$ is

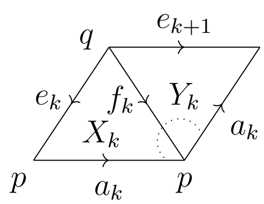
$$\smile : \bigoplus_{i=1}^g \mathbb{Z}_2 \langle \alpha_i \rangle \times \bigoplus_{i=1}^g \mathbb{Z}_2 \langle \alpha_i \rangle \to \mathbb{Z}_2 \langle \gamma \rangle \qquad \alpha_i \smile \alpha_j = \delta_{ij} \gamma$$

Proof. We define the shorthand $H^k = H^k(N_g; \mathbb{Z}_2)$. We want to understand the cup product $\smile: H^k \times H^\ell \to H^{k+\ell}$. We know that $\smile$ is unital, so the single nonzero element of $H^0(N_g; \mathbb{Z}_2)$ is the unit, so for each ℓ the product $H^0 \times H^\ell \to H^\ell$ is either the identity action on H^ℓ or the trivial action on H^ℓ. If $k + \ell \geq 3$, then the product is always zero since then $H^{k+\ell} = 0$, so the only remaining product to understand is $H^1 \times H^1 \to H^2$.

We can draw N_g as the following identification space. We draw a regular $2g$-gon and identify successive sides. For example, N_3 can be depicted as below.



We further subdivide N_g to put a Δ-complex structure on it. Add another vertex at the center of the $2g$-gon, call the central vertex q and the outer vertex p. Add radial edges from the center vertex to each outer vertex, oriented outward. We'll label these edges as e_i, f_i and the resulting triangles by X_i, Y_i so that we get the following picture.



As in Hatcher Exercise 3.8, the dotted line represents the element $\alpha_k \in H^2$, which is dual to a_k. Note that the set of α_k for $1 \leq k \leq g$ forms a basis for H^2. Note that $\alpha_i(e_k) = 0$ for all i, k, and $\alpha_i(f_j) = \delta_{ij}$, and $\alpha_i(a_j) = \delta_{ij}$. To determine $\alpha_i \smile \alpha_j$, we need to see how they act on the triangles X_k, Y_k, using the definition on page 206 of Hatcher. We can compute that

$$\begin{aligned}(\alpha_i \smile \alpha_j)(X_k) &= \alpha_i(e_k)\alpha_j(\alpha_k) = 0\\ (\alpha_i \smile \alpha_j)(Y_k) &= \alpha_i(f_k)\alpha_j(a_k) = \delta_{ik}\delta_{jk} = \delta_{ijk}\end{aligned}$$

This tells us that $\alpha_i \smile \alpha_j = 0$ if $i \neq j$. When $i \neq j$, $\alpha_i \smile \alpha_j \neq 0$, so it must be the one nonzero element of H^2. Thus we have fully understood the cup product structure (at least algebraically) on $H^k(N_g; \mathbb{Z}_2)$.

$\square$

Proposition 0.3 (Exercise 3.1.3). *As a $\mathbb{Z}_4$-module, a free resolution of $\mathbb{Z}_2$ is*

$$\dots \xrightarrow{1 \mapsto 2} \mathbb{Z}_4 \xrightarrow{1 \mapsto 2} \mathbb{Z}_4 \xrightarrow{1 \mapsto 2} \mathbb{Z}_4 \xrightarrow{1 \mapsto 1} \mathbb{Z}_2 \longrightarrow 0$$

Consequently, $\operatorname{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$ *for all $n \geq 0$. (Notably, it is nonzero for all n.)*

Proof. First, we need to show that the depicted sequence of maps is a free resolution. Obviously, $\mathbb{Z}_4$ is a free $\mathbb{Z}_4$-module, so we just need to check exactness. Note that a $\mathbb{Z}_4$-module homomorphism is determined uniquely by the image of 1, so all of our maps are well-defined $\mathbb{Z}_4$ homomorphisms. The map $\mathbb{Z}_4 \to \mathbb{Z}_2, 1 \mapsto 1$ is surjective, so we have exactness at $\mathbb{Z}_2$. At each $\mathbb{Z}_4$, the kernel is $\langle 2 \rangle$, and the image is also $\langle 2 \rangle$, so the sequence is exact. Thus this is a free resolution of $\mathbb{Z}_2$. To compute $\operatorname{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2)$, we form the deleted resolution

$$\dots \xrightarrow{1 \mapsto 2} \mathbb{Z}_4 \xrightarrow{1 \mapsto 2} \mathbb{Z}_4 \xrightarrow{1 \mapsto 2} \mathbb{Z}_4 \longrightarrow 0$$

and apply the functor $\operatorname{Hom}_{\mathbb{Z}_4}(-, \mathbb{Z}_2)$, resulting in the following chain complex.

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2) \longrightarrow \operatorname{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2) \longrightarrow \operatorname{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2) \longrightarrow \dots$$

Let $\phi : \mathbb{Z}_4 \to \mathbb{Z}_4$ denote $1 \mapsto 2$. Then the maps in the above chain complex are all the map $\psi \mapsto \psi \circ \phi$. We know that $\operatorname{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2) \cong \mathbb{Z}_2$, with the identity map corresponding to 1 and the zero map corresponding to 0. For any $\psi \in \operatorname{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2)$, the composition $\psi \circ \phi$ is zero, so our chain complex looks like

$$0 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \dots$$

That is, $\operatorname{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2)$ is the nth homology of the above chain complex. Since all them maps are zero, the nth homology is just the nth group in the complex, so $\operatorname{Ext}_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$ for all $n \geq 0$. □

Note: We have place 3.1.8a after 3.1.8b because the solution to part (a) uses the solution to part (b).

Proposition 0.4 (Exercise 3.1.8b)**.** *Let $A \subset X$ be a closed subspace that is a deformation retract of some neighborhood. Then the quotient map $q : X \to X/A$ induces isomorphisms $q_* : H^n(X, A; G) \to \widetilde{H}^n(X/A; G)$ for all n.*

Proof. Let V be a neighborhood of A in X that deformation retracts onto A. Note that $H^n(A, A) = 0$, and a deformation retraction of V onto A gives a homotopy equivalence of pairs $(V, A) \simeq (A, A)$. Thus $H^n(V, A) \cong H^n(A, A)$ by homotopy invariance, and so $H^n(V, A) = 0$. Thus in the long exact sequence of the triple (X, V, A) we obtain isomorphisms $H^n(X, A) \cong H^n(X, V)$.

Similarly, a deformation retraction of V onto A induces a deformation retraction of V/A onto A/A, so we get a homotopy equivalence of pairs $(V/A, A/A) \simeq (A/A, A/A)$, and thus $H^n(V/A, A/A) \cong H^n(A/A, A/A) = 0$. Considering the long exact sequence of the triple $(X/A, V/A, A/A)$, we get isomorphisms $H^n(X/A, A/A) \cong H^n(X/A, V/A)$.

If we think of $q : X \to X/A$ as a map of pairs $q : (X, A) \to (X/A, A/A)$ and alternately $q : (X, V) \to (X/A, V/A)$, we can fit q_* and our isomorphisms into the following diagram.

$$\begin{tikzcd} H^n(X, A) \arrow[r] \arrow[d, "q_*"] & H^n(X, V) \arrow[d, "q_*"] \\ H^n(X/A, A/A) \arrow[r] & H^n(X/A, V/A) \end{tikzcd}$$

The diagram commutes by naturality of the long exact sequence. By the Excision Theorem for cohomology, we obtain another commutative square, where the horizontal arrows are isomorphisms.

$$\begin{tikzcd}
H^n(X,V) \arrow[d, "q_*"] & H^n(X-A,V-A) \arrow[l] \arrow[d, "q_*"] \\
H^n(X/A,V/A) & H^n(X/A-A/A,V/A-A/A) \arrow[l]
\end{tikzcd}$$

The restriction $q:(X-A)\to(X/A-A/A)$ is a homeomorphism, so the restricted map of pairs $q:(X-A,V-A)\to(X/A-A/A,V/A-A/A)$ is also a homeomorphism, so the q_* on the right of the above diagram is an isomorphism. Stiching our two commutative squares together, we obtain the commutative diagram

$$\begin{tikzcd}
H^n(X,A) \arrow[r] \arrow[d, "q_*"] & H^n(X,V) \arrow[d, "q_*"] & H^n(X-A,V-A) \arrow[l] \arrow[d, "q_*"] \\
H^n(X/A,A/A) \arrow[r] & H^n(X/A,V/A) & H^n(X/A-A/A,V/A-A/A) \arrow[l]
\end{tikzcd}$$

where we now know that every horizontal map and the far right map are isomorphisms. Then by commutativity, both of the other vertical q_* maps are isomorphisms. Finally, note that $\widetilde{H}^n(X/A;G)\cong H^n(X/A,A/A)$. ☐

Proposition 0.5 (Exercise 3.1.8a). *Let G be an abelian group. Then for $n\geq 1$,*

$$\widetilde{H}^k(S^n;G)\cong\begin{cases} G & k=n \\ 0 & else\end{cases}$$

We compute this in two ways: first via the long exact sequence of a pair, and second through the Mayer-Vietoris sequence.

Proof. We will do this by induction on n, first using the LES of a pair, and then with Mayer-Vietoris. The base case applies to both, so we only do it once. For $n=0$, we know that $\widetilde{H}^k(S^0)=0$ since S^0 is a disjoint union of two discrete points.

Now assume the result holds for $S^1,\dots,S^n$. Consider the good pair (D^n,S^{n-1}). Since D^n is contractible, $\widetilde{H}^k(D^n)=0$ for all k, so the long exact sequence of the pair (D^n,S^{n-1}) breaks up into short exact sequences for each k as follows.

$$0\longrightarrow\widetilde{H}^k(D^n,S^{n-1};G)\longrightarrow\widetilde{H}^{k+1}(S^{n+1};G)\longrightarrow 0$$

By the part (b) above, since (D^n,S^{n-1}) is a good pair, we have $\widetilde{H}^k(D^n,S^{n-1};G)\cong\widetilde{H}^k(D^n/S^{n+1};G)$. Notice that $D^n/S^{n-1}=S^n$, so we obtain $\widetilde{H}^{k+1}(S^{n+1};G)\cong\widetilde{H}^k(S^n;G)$, completing the induction.

Now we repeat the same inductive step using the Mayer-Vietoris sequence, for didactic purposes. Assume the result holds for $S^1,\dots,S^n$. Let $A,B\subset S^{n+1}$ respectively be the northern and southern hemispheres, so $A \cup B = S^{n+1}$ and $A \cap B = S^n$. Part of the reduced Mayer-Vietories sequence for this decomposition of S^{n+1} is

$$\widetilde{H}^k(A;G) \oplus \widetilde{H}^k(B;G) \to \widetilde{H}^k(A \cap B;G) \to \widetilde{H}^{k+1}(S^{n+1}, G) \to \widetilde{H}^{k+1}(A;G) \oplus \widetilde{H}^{k+1}(B;G)$$

Since A, B are retractable, $\widetilde{H}^k(A;G) = \widetilde{H}^k(B;G) = 0$, so this gives an isomorphism $\widetilde{H}^k(S^n;G) \to \widetilde{H}^{k+1}(S^{n+1};G)$. Thus the reduced homology groups for $\widetilde{S}^{n+1}$ are as claimed, and the induction is complete. □

Proposition 0.6 (Exercise 3.1.8c)**.** *If a subspace A is a retract of X, then $H^n(X;G) \cong H^n(A;G) \oplus H^n(X,A;G)$. ($G$ is an arbitrary abelian group.)*

Proof. Let $r : X \to A$ be a retraction. Then we have the following commutative diagram, and the induced commmutative diagram on cohomology with coefficients in G.

$$\begin{tikzcd}
 & X \arrow[dr, "r"] & \\
A \arrow[ur, "\iota"] \arrow[rr, "\mathrm{Id}_A"'] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & H^n(X;G) \arrow[dl, "\iota^*"'] & \\
H^n(A;G) & & H^n(A;G) \arrow[ul, "r^*"'] \arrow[ll, "\mathrm{Id}_A^*=\mathrm{Id}"]
\end{tikzcd}$$

In particular, since ι_* has a right inverse, it is surjective. Consider the long exact sequence on cohomology of the pair (X, A).

$$\ldots \xrightarrow{\iota^*} H^{n-1}(A;G) \xrightarrow{\delta} H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{\iota^*} H^n(A;G) \xrightarrow{\delta} \ldots$$

From this long exact sequence, we obtain the following short exact sequence.

$$0 \longrightarrow H^n(X,A;G)/\operatorname{im}\delta \xrightarrow{j^*} H^n(X;G) \xrightarrow{\iota^*} \operatorname{im}\iota_* \longrightarrow 0$$

As already noted, ι_* is surjective, so $\operatorname{im}\iota_* = H^n(A;G)$. Since ι_* is surjective, by exactness at $H^{n-1}(A;G)$, $\ker\delta = H^{n-1}(A;G)$, that is, δ is the zero map. Thus $\operatorname{im}\delta = 0$. Then the short exact sequence becomes

$$0 \longrightarrow H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{\iota^*} H^n(A;G) \longrightarrow 0$$

As already noted, the induced map $r_* : H^n(A;G) \to H^n(X;G)$ composes with ι_* to give the identity on $H^n(A;G)$. That is to say, this short exact sequence splits, so by the splitting lemma we get the desired direct sum decomposition.

$$H^n(X;G) \cong H^n(A;G) \oplus H^n(X,A;G)$$

□

Proposition 0.7 (Exercise 3.1.9)**.** *Suppose $f : S^n \to S^n$ has degree d. Then $f^* : H^n(S^n;G) \to H^n(S^n;G)$ is multiplication by d. (Recall that $H^n(S^n;G) \cong G$.)*

Proof. We follow the proof of Lemma 2.49 from Hatcher. For $g \in G$, let $\phi_g : \mathbb{Z} \to G$ be the map $1 \mapsto g$. Note that ϕ_g induces a map on the cochain complexes $(\phi_g)_* : C_n^*(S^n;\mathbb{Z}) \to C_n^*(S^n;G)$, which acts on a general cochain as

$$(\phi_g)_* \left(\sum_i n_i \alpha_i \right) = \sum_i (n_i g) \alpha_i$$

where $n_i \in \mathbb{Z}$ and $\sum_i n_i \alpha_i$ is a singular cochain. Since $(\phi_g)_*$ commutes with the coboundary operator, it induces a map $(\phi_g)_* : \widetilde{H}^n(S^n;\mathbb{Z}) \to \widetilde{H}^n(S^n;G)$. Since f^* is $\mathbb{Z}$-linear, that is,

$$f^* \left(\sum_i n_i \alpha_i \right) = \sum_i n_i f^*(\alpha_i)$$

we get that f^* "commutes" with $(\phi_g)_*$ by a relatively simple calculation. (We put commutes in quotations because f^* refers to two different maps on either side of the equal sign.)

$$\begin{aligned}
(\phi_g)_* \circ f^* \left(\sum_i n_i \alpha_i \right) &= (\phi_g)_* \left(\sum_i n_i f^*(\alpha_i) \right) = \sum_i (n_i g) f^*(\alpha_i) \\
&= f^* \left(\sum_i (n_i g) \alpha_i \right) = f^* \circ (\phi_g)_* \left(\sum_i n_i \alpha_i \right) \\
\implies (\phi_g)_* \circ f^* = f^* \circ (\phi_g)_* &
\end{aligned}$$

Thus the middle square of the following diagram commutes.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\cong"] \arrow[d, "\phi_g"] & \widetilde{H}^n(S^n;\mathbb{Z}) \arrow[r, "f^*"] \arrow[d, "(\phi_g)_*"] & \widetilde{H}^n(S^n;\mathbb{Z}) \arrow[r, "\cong"] \arrow[d, "(\phi_g)_*"] & \mathbb{Z} \arrow[d, "\phi_g"] \\
G \arrow[r, "\cong"] & \widetilde{H}^n(S^n;G) \arrow[r, "f^*"] & \widetilde{H}^n(S^n;G) \arrow[r, "\cong"] & G
\end{tikzcd}$$

Thinking about the formula for $(\phi_g)_*$ on cochains above, we see that the outer squares commute. Since the upper f^* is multiplication by d (by hypothesis), the lower f^* must also be be multiplication by d in order to satisfy commutativity. $\square$

Lemma 0.8 (for Exercise 3.1.13)**.** *Let X, Y be path connected and $f : (X, x_0) \to (Y, y_0)$ be continuous. Let $p_X : \pi_1(X) \to H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)]$ be the canonical projection. Define p_Y by analogy. Let f_*^π be the induced map on π_1 and let f_*^H be the induced map on H^1. Then we have the following commutative diagram.*

$$\begin{tikzcd}
\pi_1(X) \arrow[r, "f_*^\pi"] \arrow[d, "p_X"] & \pi_1(Y) \arrow[d, "p_Y"] \\
H_1(X) \arrow[r, "f_*^H"] & H_1(Y)
\end{tikzcd}$$

Proof. Let $[\gamma] \in \pi_1(X)$, with representative $\gamma : S^1 \to X$. Then we have the following commutative diagrams.

$$\begin{tikzcd} S^1 \arrow[d, "\gamma"'] \arrow[dr, "f\circ\gamma"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} H_1(S^1) \arrow[d, "\gamma_*^H"'] \arrow[dr, "(f\circ\gamma)_*^H"] & \\ H_1(X) \arrow[r, "f_*^H"'] & H_1(Y) \end{tikzcd}$$

Commutativity of the left diagram is obvious, commutativity of the second is just the statement that H_1 is a covariant functor. Using the discussion on page 168 of Hatcher, we have a good way to think about p_X and p_Y. We know that $H^1(S^1) \cong \mathbb{Z}$; let a be the generator. As noted by Hatcher, an equivalent definition of p_X and p_Y is

$$p_X[\gamma] = \gamma_*^H(a) \qquad p_Y[\eta] = \eta_*^H(a)$$

Now we can just do a computation to see that the original square commutes. Recall the the definition of f_*^π is just $f_*^\pi[\gamma] = [f \circ \gamma]$.

$$f_*^H \circ p_X[\gamma] = f_*^H \circ \gamma_*^H(a) = (f \circ \gamma)_*^H(a) = p_Y[f \circ \gamma] = p_Y \circ f_*^\pi[\gamma]$$

Thus our square commutes. $\square$

Lemma 0.9 (for Exercise 3.1.13)**.** *Let X be path connected and G an abelian group. Let K be a $K(G,1)$ space. Let $p = p_X$ and f_*^H, f_*^π be as in the previous lemma. Then we have the following commutative diagram.*

$$\begin{tikzcd} \pi_1(X) \arrow[d, "p"'] \arrow[dr, "f_*^\pi"] & \\ H_1(X) \arrow[r, "f_*^H"'] & G \end{tikzcd}$$

Proof. Since G is abelian, $G \cong \pi_1(K) \cong H_1(K)$, so the right side of the square from the previous lemma collapses. $\square$

Definition 0.1. Let (X, x_0) and (Y, y_0) be pointed spaces. The we define $\langle \boldsymbol{X}, \boldsymbol{Y} \rangle$ to be the set of basepoint-preserving homotopy classes of basepoint-preserving maps $X \to Y$.

Proposition 0.10 (Exercise 3.1.13)**.** *Let (X, x_0) be a connected CW complex, and G be an abelian group. Then define a map*

$$\Phi : \langle X, K(G,1) \rangle \to \mathrm{Hom}(H_1(X), G) = H^1(X; G)$$

For $f : X \to K(G,1)$, let $[f]$ denote the class of f up to basepoint-preserving homotopy, and define $\Phi[f] = f_*^H : H_1(X) \to H_1(K(G,1)) \cong G$. (Recall that $\mathrm{Hom}(H_1(X), G) = H^1(X; G)$ via the UCT.) Then Φ is a bijection.

Proof. First, note that Φ is well defined because the induced map f_*^H depends only on the homotopy class of f.

In the following, we suppress basepoints in all of our notation, but the reader should keep in mind that X and $K(G,1)$ do have basepoints.

As in the previous lemmas, let $p : \pi_1(X) \to H_1(X)$ be the canonical projection. We define $\Psi : \mathrm{Hom}(H_1(X), G) \to \langle X, K(G,1) \rangle$ as follows. Given $\alpha \in \mathrm{Hom}(H_1(X), G)$, consider the composition $\alpha \circ p : \pi_1(X) \to G$.

$$\begin{tikzcd} \pi_1(X) \arrow[d, "p"'] \arrow[dr, "\alpha\circ p"] & \\ H_1(X) \arrow[r, "\alpha"'] & G \end{tikzcd}$$

Since $G = \pi_1(K(G,1))$, by Proposition 1.B9, there is a basepoint-preserving map $\widetilde{\alpha} : X \to K(G,1)$ inducing $\alpha \circ p$ (that is, $\widetilde{\alpha}_*^\pi = \alpha \circ p$), and furthermore $\widetilde{\alpha}$ is unique up to homotopy fixing x_0. Then we define $\Psi(\alpha) = [\widetilde{\alpha}]$. Thus it is clear that $[\widetilde{\alpha}] \in \langle X, K(G,1) \rangle$. However, we still need to check that Ψ is well-defined.

To check that Ψ is well-defined, we need to check that if $\alpha = \beta$, then $\Psi(\alpha) = \Psi(\beta)$. If $\alpha = \beta$, then $\alpha \circ p = \beta \circ p$. Then by Proposition 1.B9, we get $\widetilde{\alpha}, \widetilde{\beta} : X \to K(G,1)$, both preserving basepoints. By the "uniqueness up to basepoint-preserving homotopy," since $\alpha \circ p = \beta \circ p$, there is a basepoint-preserving homotopy $\widetilde{\alpha} \simeq \widetilde{\beta}$. Thus $[\widetilde{\alpha}] = [\widetilde{\beta}]$, so $\Psi(\alpha) = \Psi(\beta)$. Thus Ψ is well-defined.

Now we claim that Ψ is inverse to Φ. Let $\alpha \in \mathrm{Hom}(H_1(X), G)$. Then

$$\Phi \circ \Psi(\alpha) = \Phi\,[\widetilde{\alpha}] = \widetilde{\alpha}_*^H$$

By the previous lemma, $\widetilde{\alpha}_*^\pi = \widetilde{\alpha}_*^H \circ p$. By definition of Ψ, $\widetilde{\alpha}_*^\pi = \alpha \circ p$. Thus $\alpha \circ p = \widetilde{\alpha}_*^H \circ p$, so $\alpha = \widetilde{\alpha}_*^H$ on the image of p. But p is surjective, so this says that $\alpha = \widetilde{\alpha}_* H$. Thus $\Phi \circ \Psi(\alpha) = \alpha$. Now let $[f] \in \langle X, K(G,1) \rangle$ and choose a representative $f : X \to K(G,1)$. Then

$$\Psi \circ \Phi[f] = \Psi(f_*^H)$$

By the previous lemma, $f_*^\pi = f_*^H \circ p$, so f is a map that induces $f_*^H \circ p$, so $\Psi(f_*^H) = [f]$. Thus $\Psi \circ \Phi[f] = [f]$. $\square$